Objectives

- Determine if two events are mutually exclusive
- Find probabilities using the Addition Rule for Mutually Exclusive Events
- Find probabilities using the Addition Rule for Probability
- Find the probability of a set and its complement
- Learn the basic properties of probabilities

Nov 8-8:12 PM

Probability Rules: Properties, the Complement, and Addition Rules

What Is Probability?

There are several competing ideas which seek to define the interpretation of probability, similar to the competing ideas which define the notion of central tendency (mean, median, mode, trimmed mean). In fact, there is a substantial conflict of ideas between the notions of statistical regularity and degree of belief. No simple answer exists for the question, "What is probability?" Probability remains an abstract concept for which there is no true meaning. Fortunately, the theory of probability does not require the interpretation of probabilities, just as in geometry the interpretation of points, lines, and planes are irrelevant.

Nov 8-8:13 PM

Some Laws of Probability

Interpreting probability using the classical approach is a good way of thinking about the basic probability principles. In this section we will discuss certain laws that probabilities must obey, regardless of how probability is defined.

Nov 8-8:14 PM

Probability Law 1

A probability of zero means the event cannot happen.

For example, the probability of observing three heads in two tosses of a coin is zero.

Nov 8-8:15 PM

Probability Law 2

A probability of one means the event must happen.

For example, if we toss a coin, the probability of getting either a head or tail is one.

Nov 8-8:16 PM

Probability Law 3

All probabilities must be between zero and one inclusively. That is, $0 \leq P(A) \leq 1$.

The closer the probability is to 1, the more likely the event. The closer the probability is to 0, the less likely the event.

Nov 8-8:16 PM

Probability Law 4

The sum of the probabilities of all outcomes in a sample space must equal one. That is, if $P(A_i)$ is the probability of outcome A_i, and there are n such outcomes, then $P(A_1) + P(A_2) + \cdots + P(A_n) = 1$.

Nov 8-8:18 PM

What's the Connection Between Probability and Statistics?

Most of the time, when working with samples, statisticians try to deduce from the samples the population parameters (means, proportions, variances, etc.) of certain variables. This process of making judgments about population parameters is called statistical inference. Because samples are random, there is no guarantee that the sample will be representative of the population. If the sample is not representative, then using the sample mean as an estimate (inference) of the population mean would not be very wise. Probability is used to assess the quality of our inference. All statistical conclusions must be endowed with a degree of uncertainty. Because probability is used to assess the reliability of sample inferences, it is the foundation of all inferential statistics.

Nov 8-8:18 PM

Probability and Business

- The probability concept also has many direct applications in business. When a manager wonders whether dropping a bid price by 5% will increase the probability of winning the bid, he or she is thinking about chance.

- Probability is also used as a criterion in designing and evaluating product reliability, evaluating insurance, inventory management, project management, and in the study of queuing theory (a probabilistic analysis of waiting lines).

Nov 8-8:19 PM

- Probability theory emerged from the need to better understand a game of chance.

- Business decisions, like games, have uncertain outcomes. In an effort to make better decisions, businesses spend considerable amounts of money trying to quantify uncertainty. This means trying to turn uncertainty into a probability.

- Insurance companies have historically done a good job of quantifying uncertainty. In fact, a special kind of statistician called an actuary has emerged to assist in the development of insurance models which quantify uncertainty and aid in business decisions.

Nov 8-8:19 PM

- For example, the next time you watch a 30-second commercial during the Super Bowl, consider the fact that a company has just spent roughly $3 million for the air time plus a substantial amount of money developing the advertisement.
- Without knowing the effect of the advertisement in advance, extensive amounts of money are put at risk with an uncertain outcome.
- The manager making the decision uses subjective probability to assess the risk and reward.

Nov 8-8:21 PM

Other Probability Rules

There are a number of rules concerning the relationships between events that are useful in determining probabilities.

Definition

A **compound event** is an event that is defined by combining two or more events.

Nov 8-8:22 PM

Suppose that the marketing director of Sports Illustrated believed that anyone who possessed an income greater than \$50,000 and/or subscribed to more than one other sports magazine could potentially be a good prospect for a direct mail marketing campaign.

Let the events

A={annual income is greater than \$50,000}

and

B={subscribes to more than one other sports magazine}

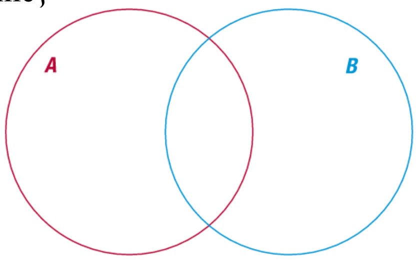


Figure 6.4

Nov 8-8:23 PM

There are several different types of compound events. To illustrate the concepts consider the two events A and B in Figure 6.4. The set of outcomes in which either or both of these events occurs is called the union of the two sets.

A U B

Definition

The **union** of the events *A* and *B* is the set of outcomes that are included in *A* or *B* or both, and is denoted $A \cup B$.

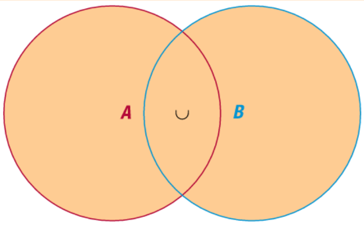
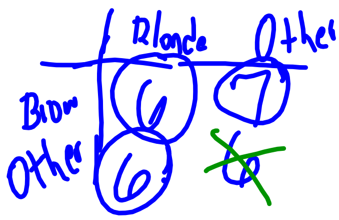


Figure 6.5

Notice in Figure 6.5 that the union includes all points in both A and/or B. The marketing director for Sports Illustrated would see this as the group of people that have an annual income greater than \$50,000 and/or subscribe to more than one other sports magazine.



Nov 8-8:25 PM

Suppose the marketing director was interested in persons who possessed an annual income greater than $50,000 and subscribed to more than one other sports magazine; that set would be called the intersection of A and B.

$A \cap B$

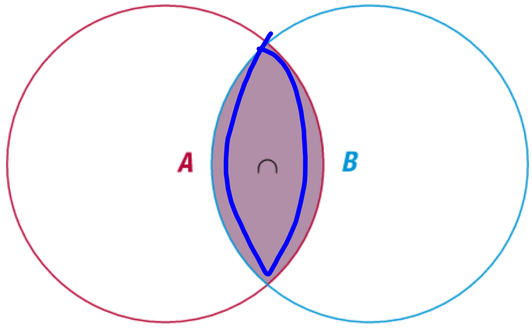


Figure 6.6

Notice in Figure 6 that the intersection includes only those points in both A and B.

Nov 8-8:29 PM

Two other useful concepts are the notions of the complement of an event and events which are mutually exclusive.

A^c A'

Definition

The **complement** of an event *A* is the set of all outcomes in the sample space that are not in *A*.

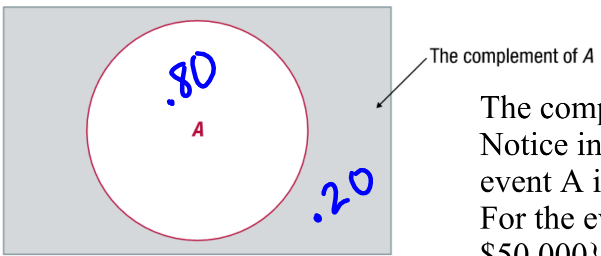


Figure 6.7

The complement of the set A is written as Ac. Notice in Figure 6.7 that the complement of event A includes all points which are not in A.A. For the event A={annual income is greater than $50,000}, the complement of AA would be

A^c={annual income is less than or equal to $50,000}.

Nov 8-8:31 PM

Also note that $A_c \cup A = S$.

Probability Law 5

The probability of A^c is given by $P(A^c) = 1 - P(A)$.

Sometimes it is much easier to calculate the probability of the complement of an event than the probability of the actual event.

$P(A) + P(A^c) = 1$
$P(A^c) = 1 - P(A)$

Nov 8-8:56 PM

Example 6.5

Consider the event A = {annual income is greater than \$50,000}. Suppose the probability of A was 0.08. Determine the probability of observing someone whose income was less than or equal to \$50,000.

$P(A) = .08$ A People > 50,000
$P(A^c) = .92$ A^c People ≤ 50,000

Solution

$$\begin{aligned} P(\text{annual income is less than or equal to \$50,000}) &= P(A^c) \\ &= 1 - P(A) \\ &= 1 - 0.08 \\ &= 0.92 \end{aligned}$$

Nov 8-8:58 PM

Example 6.6

Consider an experiment to see how many tosses of a coin will be required to obtain the first head. The first head could be observed on the first toss or the second toss but there is no upper limit on the number of tosses that could be required. Therefore, the sample space for this experiment is the set of positive integers $\{1, 2, 3, \ldots\}$. Not only is the sample space infinitely large, but there is another problem: the outcomes are not equally likely. This is a potentially ugly environment in which to compute a probability. But let's make matters slightly worse. Suppose we want to know the probability that it will require at least two tosses to get the first head. That is, we want to know the probability of getting the first head in two or more tosses of the coin. This means that we must compute the probability of 2, the probability of 3, and so on up to infinity and add them up in some way. The problem is rather insidious if approached directly.

P(at least 2 tosses to get 1 head) A

P(head on first) is A^c

Nov 8-8:59 PM

Solution

The problem becomes rather trivial by determining the complement of the event and computing its probability. The complement of obtaining the first head in two or more tosses is getting a head on the first toss. The probability of getting a head on the first toss is 0.5, assuming the coin is fair. Therefore, we have the following.

$$\begin{aligned} P(\text{two or more tosses to obtain the first head}) &= 1 - P(\text{head on the first toss}) \\ &= 1 - 0.5 \\ &= 0.5 \end{aligned}$$

As shown in this example, when you see the key words "at least" in a probability problem, you will often want to use the complement to find the appropriate probability.

Nov 8-9:00 PM

Another idea that is helpful in determining probabilities is the notion of mutual exclusivity.

Definition

Two events are **mutually exclusive** if they have no outcomes in common.

Mutual exclusivity is also called disjointedness. Figure 6.8 represents two disjoint events.

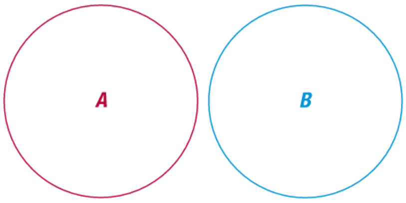


Figure 6.8

Nov 8-9:01 PM

Mutually Exclusive

Two events are mutually exclusive if they cannot occur at the same time.

For example, if you were to select one card from a standard deck, the two outcomes

- The card is a Jack
- The card is a Seven

are mutually exclusive, because you cannot select a card that is both a Jack *and* a Seven.

However, the two outcomes

- The card is a Jack
- The card is a Club

are not mutually exclusive, because you can select a card that is both a Jack *and* a Club.

Nov 8-9:03 PM

Example 6.7

Suppose $P(A) = 0.27$ and $P(B) = 0.19$. If A and B are mutually exclusive, what is the probability of $A \cup B$?

Solution

Since these are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B) = 0.27 + 0.19 = 0.46.$$

Nov 8-9:04 PM

There is a more generalized rule that eliminates the assumption of mutual exclusivity between the sets.

Probability Law 8: The Addition Rule

For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

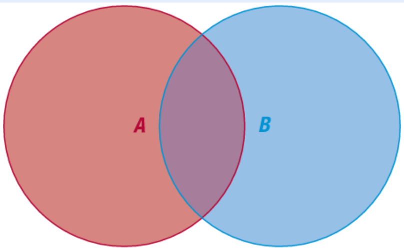


Figure 6.9

The addition rule is identical to Probability Law 6 when the events are mutually exclusive, since A intersect B will be an empty set whose probability will be zero (i.e., $P(A \cap B)=0$).

Nov 8-9:23 PM

Example 6.8

Suppose that the marketing manager mentioned earlier believed that the probability that someone earns more than $50,000 is 0.2 and the probability that someone subscribes to more than one sports magazine is 0.3. If the probability of finding someone in both categories is 0.08, what is the probability of finding someone who is earning over $50,000 or subscribes to more than one sports magazine, or both?

Solution

The problem involves the union of two events. Using the same event names (A and B) as in previous examples, the desired probability is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.2 + 0.3 - 0.08 = 0.42.$$

Therefore, the probability of finding someone who is earning over $50,000 or subscribes to more than one sports magazine, or both is 0.42.

Nov 8-9:26 PM