Objectives

- Combine previously encountered techniques to solve more complex applications

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Additional Counting Techniques

Combining Probability and Counting Techniques

In previous sections, we have looked at techniques for solving simple counting problems. In this section, we will now introduce an assortment of counting problems that have more complicated solutions. A helpful trick for these examples is to look for certain key terms. The key terms are important because they define the method that must be used in order to obtain the correct answer. Words to look for include at least, at most, greater than, less than, between, and so forth. A few examples of the many different types of problems you may encounter are given below.

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Example 6.22: Calculating Probability Using Combinations

A group of 1212 tourists is visiting London. At one particular museum, a discounted admission is given to groups of at least ten.

1. How many combinations of tourists can be made for the museum visit so that the group receives the discounted rate?
2. Suppose that a group of the tourists does get the discount. What's the probability that it was made up of 1111 of the tourists?

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Solution

1. The key words for this problem are at least. If at least 10 are required, then the group will get the discount rate if 10, 11, or 12 tourists go to the museum. We must then calculate the number of combinations for each of the three possibilities. Note that we are counting combinations, not permutations, because the order of tourists chosen to go to the museum is irrelevant.

#	10	11	12
	$_{12}C_{10} = \frac{12!}{10!(12-10)!}$	$_{12}C_{11} = \frac{12!}{11!(12-11)!}$	$_{12}C_{12} = \frac{12!}{12!(12-12)!}$
	$= \frac{12!}{10!2!}$	$= \frac{12!}{11!1!}$	$= \frac{\cancel{12!}}{\cancel{12!}\,0!}$
	$= \frac{\overset{6}{\cancel{12}} \cdot 11 \cdot \cancel{10 \cdot 9 \cdots 2 \cdot 1}}{\cancel{(10 \cdot 9 \cdots 2 \cdot 1)}(\cancel{2} \cdot 1)}$	$= \frac{12 \cdot \cancel{11 \cdot 10 \cdots 2 \cdot 1}}{\cancel{(11 \cdot 10 \cdots 2 \cdot 1)}(1)}$	$= \frac{1}{1}$
	$= 6 \cdot 11$	$= 12$	$= 1$
	$= 66$		

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As we noted, the problem implies that the group will get the discount rate if 10, 11, or 12 tourists attend. Remember, the word "or" tells us to add the results together. So, to find the total number of groups, we add the numbers of combinations together.

$$66+12+1=79$$

There are 79 different groups that can be formed to tour the museum at the discounted rate.

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1. To calculate this probability, we need to think of this as a conditional probability. Think of the sentence as reading, "Given that a group of tourists received the discount, what's the probability that it was a group of 11?" So, to find the probability, we need the total number of ways a group of 11 can be chosen from the group of 12 divided by the number of ways that a group could receive the discount. Since both of these were calculated in part a., we can simply substitute these numbers in our fraction. Therefore, we calculate the probability as follows.

$$P(11 \text{ tourists} \mid \text{discount}) = \frac{12}{79} \approx 0.1519$$

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Example 6.23: Calculating Probability Using Permutations

Jack is setting a password on his computer. He is told that his password must contain at least three, but no more than five, characters. He may use either letters or numbers (0-90-9).

1. How many different possibilities are there for his password if each character can only be used once?
2. Suppose that Jack's computer randomly sets his password using all of the restrictions given above. What is the probability that this password would contain only the letters in his name?

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1. First, notice the key words at least and no more than. These words tell us that Jack's password can be 3, 4, or 5 characters in length. Also note that for a password, the order of characters is important, so we are counting permutations. There are 26 letters and 10 digits to choose from, so we must calculate the number of permutations possible from taking groups of 3, 4, or 5 characters out of 36 possibilities.

Characters: 3

$$
\begin{aligned}
{}_{36}P_3 &= \frac{36!}{(36-3)!} \\
&= \frac{36!}{33!} \\
&= \frac{36\cdot35\cdot34\cdot \cancel{33\cdot32\cdots2}}{\cancel{33\cdot32\cdots2\cdot1}} \\
&= 36\cdot35\cdot34 \\
&= 42{,}840
\end{aligned}
$$

4

$$
\begin{aligned}
{}_{36}P_4 &= \frac{36!}{(36-4)!} \\
&= \frac{36!}{32!} \\
&= \frac{36\cdot35\cdot34\cdot33\cdot \cancel{32\cdot31}}{\cancel{32\cdot31\cdots2\cdot1}} \\
&= 36\cdot35\cdot34\cdot\vdots \\
&= 1{,}413{,}720
\end{aligned}
$$

5

$$
\begin{aligned}
{}_{36}P_5 &= \frac{36!}{(36-5)!} \\
&= \frac{36!}{31!} \\
&= \frac{36\cdot35\cdot34\cdot33\cdot32\cdot \cancel{31\cdot30\cdots2\cdot1}}{\cancel{31\cdot30\cdots2\cdot1}} \\
&= 36\cdot35\cdot34\cdot33\cdot32 \\
&= 45{,}239{,}040
\end{aligned}
$$

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To find the total number of possible passwords, once again we need to add all three of the above numbers of permutations together since the solution contains an "or" statement. Therefore, we obtain 46,695,600 passwords.

42,840+1,413,720+45,239,040=46,695,600

(Imagine how many possibilities there would be if he could use up to eight or nine characters!)

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1. To find the probability that a randomly chosen password would include only the four letters from Jack's name, we need the total number of possible passwords calculated in part a. as well as the number of possible passwords made from the four letters in Jack's name. To find this second number, we need to calculate the number of permutations of 44 things from a set of 4.

$$
\begin{aligned}
{}_4P_4 &= \frac{4!}{(4-4)!} \\
&= \frac{4 \cdot 3 \cdot 2 \cdot 1}{0!} \\
&= \frac{24}{1} \\
&= 24
\end{aligned}
$$

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So, the probability is calculated as follows.

$$P(\text{password containing only the letters J, A, C, K}) = \frac{24}{46{,}695{,}600} \approx 0.0000005$$

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Combining Counting Techniques

At times it is necessary to combine counting techniques in order to find all possibilities for a given problem. A common scenario that occurs is when a combination or permutation formula must be used in order to determine the number of outcomes for each slot of a Fundamental Counting Principle problem. Solving this type of problem is best learned by example.

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Example 6.24: Using Numbers of Combinations with the Fundamental Counting Principle

Tina is packing her suitcase to go on a weekend trip. She wants to pack 33 shirts, 22 pairs of pants, and 2 pairs of shoes. She has 99 shirts, 55 pairs of pants, and 4 pairs of shoes to choose from. How many ways can Tina pack her suitcase? (We will assume that everything matches.)

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Solution

Begin by thinking of this problem as a Fundamental Counting Principle problem. Consider the act of packing the suitcase to be a multistage experiment. There are three stages—packing the shirts, packing the pants, and packing the shoes. Thus, there are three suitcase "slots" for Tina to fill—a slot for shirts, a slot for pants, and a slot for shoes. To fill these slots, the combination formula must be used because we are choosing some items out of her closet, and the order of the choosing does not matter. Let's first calculate these individual numbers of combinations.

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Shirts	Pants	Shoes
$_9C_3 = \frac{9!}{3!(9-3)!}$	$_5C_2 = \frac{5!}{2!(5-2)!}$	$_4C_2 = \frac{4!}{2!(4-2)!}$
$= \frac{9!}{3!6!}$	$= \frac{5!}{2!3!}$	$= \frac{4!}{2!2!}$
$= \frac{\overset{3}{\cancel{9}} \cdot \overset{4}{\cancel{8}} \cdot 7 \cdot \cancel{6 \cdot 5 \cdots 2 \cdot 1}}{(\cancel{3} \cdot \cancel{2} \cdot 1)\ \cancel{(6 \cdot 5 \cdots 2 \cdot 1)}}$	$= \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3 \cdot 2 \cdot 1}}{(\cancel{2} \cdot 1)\ \cancel{(3 \cdot 2 \cdot 1)}}$	$= \frac{\overset{2}{\cancel{4}} \cdot 3 \cdot \cancel{2 \cdot 1}}{(\cancel{2} \cdot 1)\ \cancel{(2 \cdot 1)}}$
$= 3 \cdot 4 \cdot 7$	$= 5 \cdot 2$	$= 2 \cdot 3$
$= 84$	$= 10$	$= 6$

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The final step in this problem is different than the final step in the last two examples, because of the word "and." The Fundamental Counting Principle tells us we have to multiply, rather than add, the numbers of combinations together to get the final result. There are then

$$84 \cdot 10 \cdot 6 = 5040$$

different ways that Tina can pack for the weekend.

Shirts		Pants		Shoes
$_9C_3$	$\times$	$_5C_2$	$\times$	$_4C_2$
84	$\times$	10	$\times$	6
		5040		

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Memory Booster

And vs. Or in Probability

"And" $\Longleftrightarrow$ Multiply

"Or" $\Longleftrightarrow$ Add

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Example 6.25: Using Numbers of Combinations with the Fundamental Counting Principle

An elementary school principal is putting together a committee of 6 teachers to head up the spring festival. There are 8 first-grade, 9 second-grade, and 7 third-grade teachers at the school.

1. In how many ways can the committee be formed?
2. In how many ways can the committee be formed if there must be 22 teachers chosen from each grade?
3. Suppose the committee is chosen at random and with no restrictions. What is the probability that 2'22 teachers from each grade are represented?

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a. There are no stipulations as to what grades the teachers are from and the order in which they are chosen does not matter, so we simply need to calculate the number of combinations by choosing 66 committee members from the 8+9+7=24 teachers.

$$
\begin{aligned}
{}_{24}C_6 &= \frac{24!}{6!(24-6)!} \\
&= \frac{24!}{6!18!} \\
&= \frac{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdots 2 \cdot 1}{(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)\ (18 \cdot 17 \cdots 2 \cdot 1)} \\
&= 23 \cdot 11 \cdot 7 \cdot 4 \cdot 19 \\
&= 134{,}596
\end{aligned}
$$

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b. For this problem we need to combine the Fundamental Counting Principle and the combination formula. First, we see that there are three slots to fill: one first-grade slot, one second-grade slot, and one third-grade slot. Each slot is made up of 22 teachers, and we must use the combination formula to determine how many ways these slots can be filled.

First

$$
\begin{aligned}
{}_8C_2 &= \frac{8!}{2!(8-2)!} \\
&= \frac{8!}{2!6!} \\
&= \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdots 2 \cdot 1}{(2 \cdot 1)\ (6 \cdot 5 \cdots 2 \cdot 1)} \\
&= 4 \cdot 7 \\
&= 28
\end{aligned}
$$

Second

$$
\begin{aligned}
{}_9C_2 &= \frac{9!}{2!(9-2)!} \\
&= \frac{9!}{2!7!} \\
&= \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdots 2 \cdot 1}{(2 \cdot 1)\ (7 \cdot 6 \cdot 5 \cdots 2 \cdot 1)} \\
&= 9 \cdot 4 \\
&= 36
\end{aligned}
$$

Third

$$
\begin{aligned}
{}_7C_2 &= \frac{7!}{2!(7-2)!} \\
&= \frac{7!}{2!5!} \\
&= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)\ (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)} \\
&= 7 \cdot 3 \\
&= 21
\end{aligned}
$$

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Finally, the Fundamental Counting Principle says that we have to multiply together the outcomes for all three slots in order to obtain the total number of ways to form the committee. Thus, there are

$$28 \cdot 36 \cdot 21 = 21{,}168$$

ways to choose the committee.

1st Grade		2nd Grade		3rd Grade
${}_8C_2$	×	${}_9C_2$	×	${}_7C_2$
28	×	36	×	21
		21,168		

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c. The final part of this question asks us to find the probability that if a committee was chosen at random, it would meet the requirements given in part b. That is, each grade would have two teachers on the committee. We can combine our answers from parts a. and b. to find the answer to this probability question.

$$P(\text{committee has 2 teachers from each grade})$$
$$= \frac{\text{Number of possible committees with 2 teachers from each grade}}{\text{Number of possible committees of 6 teachers}}$$
$$= \frac{21{,}168}{134{,}596}$$
$$\approx 0.1573$$

In other words, if the members of the committee are randomly selected, there is a 15.73% chance that the committee will be made up of 2 teachers from each grade.

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