Objectives

- Evaluate expressions containing factorials
- To differentiate between permutation and combination
- To learn the five basic counting rules

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Basic Counting Rules

To compute certain probabilities, such as the probability of having winning numbers in the state lottery, requires the ability to count the number of possible outcomes for a given experiment or a sequence of experiments.

However, often it is impractical to list out all the possibilities. Therefore, we will develop some techniques to facilitate our counting.

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The Fundamental Counting Principle

Theorem: Fundamental Counting Principle

E_1 is an event with n_1 possible outcomes and E_2 is an event with n_2 possible outcomes. The number of ways the events can occur in sequence is $n_1 \cdot n_2$. This principle can be applied for any number of events occurring in sequence.

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Example 6.14

A local office supply store offers ballpoint pens from three different manufacturers. Each manufacturer's pens come in either red, blue, black, or green and either fine or medium tip is available for each color. How many different pens does the store carry?

Solution

$$\underset{\left(\begin{array}{c}\text{number of}\\ \text{manufacturers}\end{array}\right)}{3} \cdot \underset{\text{(colors of ink)}}{4} \cdot \underset{\text{(types of tips)}}{2} = \underset{\text{(different pens)}}{24}$$

Thus, the store carries 24 different ballpoint pens.

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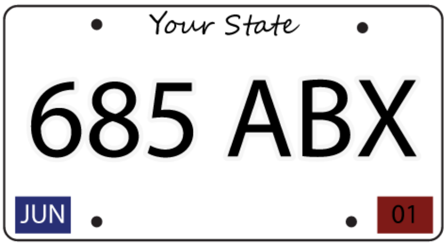
Suppose nonpersonalized license plates in your state consist of three numbers followed by three letters (excluding I, O, and Q). How many license plates are possible?

Solution

There are ten digits (0–9) possible for each of the first three characters. Likewise, there are 23 letters possible for the last three characters. Therefore, we have the following.

10	·	10	·	10	·	23	·	23	·	23	=	12,167,000
(digit)		(digit)		(digit)		(letter)		(letter)		(letter)		(possible license plates)



Therefore, there are 12,167,000 possible nonpersonalized license plates.

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Example 6.16

You have a stack of 5 textbooks: English, History, Statistics, Geology, and Psychology. How many ways can you arrange these textbooks on a shelf?

Solution

Because you can put each book on the bookshelf only once, you have five possible choices for the first book. Similarly, there are four possible choices for the second book, three possible choices for the third book, two possible choices for the fourth book, and only one book left for the fifth book. Using the Fundamental Counting Principle, we have

$$(5)(4)(3)(2)(1) = 120 \text{ possible arrangements.}$$

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Factorials

The product (5)(4)(3)(2)(1)(5)(4)(3)(2)(1) in Example 6.16 is a special type of product called a factorial. Factorials occur so frequently that they have their own notation as follows.

Formula: Factorial

Suppose n is a positive whole number. Then,

$$n! = n(n-1)(n-2)\cdots(3)(2)(1).$$

Note: $0! = 1$ by definition.

n! is read as n factorial. Note from the previous example that n! represents the number of ways to arrange n items.

Using this notation, 5!=(5)(4)(3)(2)(1)=120.

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Permutations

We have seen how the Fundamental Counting Principle and factorial notation can help us when counting ordered arrangements. These ordered arrangements are called permutations.

Definition

A **permutation** is a specific order or arrangement of objects in a set. There are $n!$ permutations of n unique objects.

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Example 6.17

To complete your holiday shopping, you need to go to the bakery, department store, grocery store, and toy store. If you are going to visit the stores in sequence, how many different sequences exist?

Solution

This is a permutation problem because order matters. By the permutation definition there are $4! = (4)(3)(2)(1) = 24$ sequences.

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Formula: Permutations

The number of permutations of n unique objects taken k at a time is

$$_nP_k = \frac{n!}{(n-k)!}.$$

Note that some alternate notations for permutations that you may see are $k!\binom{n}{k}$, P^n_k, and $P(n, k)$. All of these denote the number of permutations of n objects taken k at a time.

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Example 6.18

At a local fast food restaurant, the door to the kitchen is secured by a five-button lock, labeled 1, 2, 3, 4, 5. To open the door, the correct three-digit code must be pushed but each button can only be pushed once. How many different codes are possible?

Solution

This is a permutation problem of 5 objects, but we are taking only 3 at a time. There are 5 buttons available for the first character in the code, 4 for the second, and 3 for the third. Therefore, there are

$$(5)(4)(3) = 60 \text{ possible codes.}$$

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Example 6.19

Seven sprinters have advanced to the final heat at a track meet. How many ways can they finish in first, second, and third place?

Solution

Because we have seven sprinters and the order (first, second, third) is important, we need to find the number of permutations of 7 objects taken 3 at a time.

$${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{(7)(6)(5)\cancel{4!}}{\cancel{4!}} = 210$$

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There are times when we are interested in finding the number of permutations where some of the objects are duplicates. For instance, consider the word EYE.

If we have interchanged the two E's, the resulting permutation is not distinguishable from the original. To count the number of distinguishable permutations, we need the following formula.

Formula: Distinguishable Permutations

If given n objects, with n_1 alike, n_2 alike, …, n_k alike, then the number of distinguishable permutations of all n objects is $\frac{n!}{(n_1!)(n_2!)(n_3!)\cdots(n_k!)}$.

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Example 6.20

How many distinguishable permutations can be made from the word *Mississippi*?

Solution

There are 11 letters in the word *Mississippi*, one M, four I's, four S's, and two P's. So there are

$$\frac{11!}{(1!)(4!)(4!)(2!)} = 34{,}650$$

distinguishable permutations of the letters in *Mississippi*.

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Combinations

We have seen how the Fundamental Counting Principle and factorial notation can help us when counting the number of possible arrangements of items. We have just looked at counting where order or arrangement mattered. But there are many situations, such as choosing winning lottery numbers, where the order of the objects is not important. If we are interested in counting the number of arrangements and order is not important, we are dealing with a combination.

Definition

A **combination** is a collection or grouping of objects where the order is not important.

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The number of combinations of n objects taken k at a time can be found as follows.

Formula: Combination

The number of combinations of n unique objects taken k at a time is

$$_nC_k = \frac{n!}{(n-k)!k!}.$$

Note that some alternate notations for combinations that you may see are $\binom{n}{k}$, C_k^n, and $C(n, k)$. All of these denote the number of combinations of n objects taken k at a time.

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Example 6.21

In the Mega Millions lottery, a player selects five different numbers from 1 to 75 (inclusive) and then another (a sixth number, called the Mega Ball) from 1 to 15 (inclusive). If the first five numbers match the player's numbers in any order along with the Mega Ball number, the player wins the jackpot.

a. What is the total number of winning combinations?

b. What is the probability of winning?

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a. First, we have to determine the number of ways of selecting 5 numbers from 75. Then, we have to multiply the number of combinations of choosing 5 numbers from 75 by 15, the number of ways you can select the Mega Ball. Thus, we have the following.

$$
\begin{aligned}
{}_{75}C_5 &= \frac{75!}{(75-5)!5!} \\
&= \frac{75!}{70!5!} \\
&= \frac{(75)(74)(73)(72)(71)\,\cancel{(70!)}}{\cancel{70!}\,5!} \\
&= \frac{(75)(74)(73)(72)(71)}{(5)(4)(3)(2)(1)} \\
&= 17{,}259{,}390
\end{aligned}
$$

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So, the total number of winning combinations is

$$15 \cdot 17{,}259{,}390 = 258{,}890{,}850.$$

b. The probability of winning with any one combination is

$$\frac{1}{258{,}890{,}850}$$

which is approximately 0.0000. The actual probability is 0.00000000386263.

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